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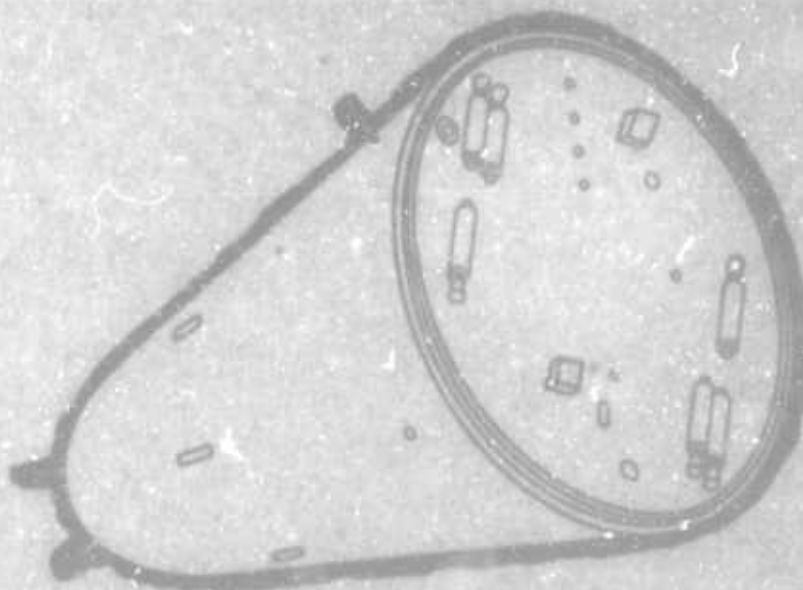# VIP EXPERIMENT CRITIQUE

FINAL REPORT NO. 6

A KINETIC THEORY MODEL FOR SHOCK FORMATION

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GENERAL ELECTRIC

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VIP EXPERIMENT CRITIQUE

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A KINETIC THEORY MODEL FOR SHOCK FORMATION

Contract No. AF 04(694)-268

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ii

TABLE OF CONTENTS

iii

I. INTRODUCTION

One of the fundamental problems in hypervelocity rarefied gasdynamics is a quantitative description of the transition between free molecule and continuum flow. The problem of providing such a description entails solving the Boltzmann transport equation subject to initial values and boundary conditions. This is, indeed, a difficult problem, and thus far analyses have predominantly dealt with simple linearized flows (Couette flow [1,2] and Rayleigh's problem [3]).

Theoretical analysis of nonlinear flows has, in general, not proceeded as successfully as for the linear case. The only nonlinear problem that has received exhaustive treatment is shock structure in a monatomic gas (an excellent review of this problem is given by Talbot [4]). The work on shock structure has, for the most part, followed the ideas of Mott-Smith [5], who considered the shock layer as the mixing zone for two Maxwellian streams. These ideas have been elaborated by various investigators [6-8] and have resulted in the postulation of a two-fluid model [6] for the shock structure problem.

The analyses of more general nonlinear flows have been influenced, to a large degree, by the theoretical models employed in the shock structure problem. Although these models are based on sound physical ideas and give reasonable results for the shock structure problem, in their application to more complicated flows one encounters serious difficulties.

Lees [9] has, for example, suggested a generalization of the Mott-Smith bimodal representation of the distribution function. This approach, although successful for compressible Couette flow, results in a set of intractable differential equations for such flows of interest as the shock structure problem. [4] Several investigators, [10,11] have also proposed variations of the multifluid, shock structure models for use in more general nonlinear, rarefied flows. Although a multifluid theory is intuitively appealing for the nonlinear regime, the investigations to date do not present a consistent formulation nor are they applied to a really representative problem. The model proposed by Rott and Whittenbury [10] assumes two fluids, a hyperthermal, "freestream" fluid and a "scattered fluid." In the model, the "freestream" fluid can be

1

conveniently represented as a molecular beam; however, the assumption that the "scattered" fluid is representable by a Maxwellian distribution function is, as pointed out by the authors, quite arbitrary. Lubonski's model,[11] on the other hand, divides the gas into three classes of particles: "freestream," "reflected from the boundary," and "scattered." Although such a classification is sensible, Ref. 11 contains little discussion on how one handles each fluid, and in fact the treatment of hypervelocity Couette flow that is given restricts considerations to the near free molecular flow regime. To summarize, one can state that although the application of the ideas used in the study of wave structure to more general problems is certainly worthwhile, there is as yet no satisfactory extension of these ideas.

In this work a new, two-fluid model for the hypervelocity rarefied regime is presented. This work avoids many of the shortcomings of the previous multifluid models and results in a set of partial differential moment equations that are of the same order of difficulty as the conventional gas dynamic equations. Although the model does not provide a description of the entire transition between the free-molecule and continuum regimes, it does yield acceptable results into the transition regime, the consistency of the simplifying assumptions being ascertainable from the numerical results.

II. MOMENT METHOD

The a priori separation of particles into "cold" and "wall" fluids in Figure 1 provides a framework for the development of a moment method for the rarefied hypervelocity regime. In examining the dynamics of the "cold" fluid, we note that far from the body the "cold" fluid is made up of hypervelocity freestream particles. As one approaches the body, the "cold" fluid begins to be populated with scattered particles (resulting from "cold-wall" intermolecular collisions) and its pressure and temperature begin to increase, i.e., it begins to thermalize. Therefore, in the hyperthermal regime we can expect that the moment equations for the cold fluid can be truncated[12] by neglecting the heat flux tensor compared to the other terms in the equation of motion for the stress tensor.

To review the rationale for the "a priori" choice of fluids we note that a "wall" fluid, encompassing those particles that are reflected from the surface and a "cold" fluid that includes all freestream and scattered particles are chosen. It will be seen that one has a rather simple description of the "wall" fluid, since it is the attenuation of the "wall" fluid density that is of most interest and not the details of the distribution function for these particles. However, in describing the termalization of the "cold" fluid, more detailed information is required because it is this thermalization that is most important in shock formation and in the dynamics of the transition regime. We first outline our treatment of the wall fluid.

For the "wall" fluid, it is expedient to adopt a much simpler description than for the cold fluid. This is because one is, for the most part, interested only in the attenuation of the "wall" fluid density as a function of distance from the body. The fluxes of energy and momentum to the body will generally be controlled by the "cold" fluid, and so one is mainly interested in the number density n_2, which interacts with the "cold" fluid through collisions. To this end we choose for f_2

$$= \frac{n_2(r, t)}{(2\pi k T_w/m)^{1/2}} \exp\left(- \frac{mv^2}{2kT_w}\right) \qquad \Omega = \Omega_b$$

$$f_2 = 0 \qquad \Omega = \Omega_c$$

(1)

where Ω_b is the solid angle that the body subtends at the point r. Because equation (1) is the free molecular distribution function for f_2, with the density n_2 an undetermined function of time and position, equation (1) will reduce to the correct free-molecule solution in the limit of Knudsen number $\to \infty$. The rationale for the choice of only one free variable in (1) is that, for $V_1^2/(2kT_w/m_1) \gg 1$, "cold-wall" collisions do not have a large effect on the form of the distribution function of the "wall" particles. One therefore considers the main effect of "cold-wall" collisions to be the attenuation of the "wall" fluid density.

For the "cold fluid" Hamel[12] has considered a truncation of the moment equations by neglect of the heat flux tensor. The difficulty he encounters in the analysis of the one-dimensional piston problem is that the moment differential equations are hyperbolic so that a shock-like discontinuity tends to form in the "cold fluid". He is therefore able to obtain results with the method which give the initial development of a shock-like structure in the "cold fluid". This can be seen in Figure 2, where we plot the "cold-fluid" density as a function of position for a piston Mach number of 10. The main criticisms of the Hamel's model are that it fails to take account of the heat flux for the region of flow where significant thermalization has taken place and in addition it neglects self-collisions among scattered particles. To overcome these difficulties here, we propose a modification of the original model, which has already proved successful in the kinetic theory analysis of expansions into vacuum[13].

To make our application of this new method clear we make the equations appropriate to the problem of one-dimensional hypersonic compression.

The problem we consider here is that of a piston that is impulsively brought to hypersonic velocity at t = 0. In the coordinate system of the piston, the gas molecules can be considered to have a hypersonic velocity $[V_\infty^2/(2kT_\infty/m) \gg 1]$, and the plate is stationary at x = 0. It is assumed that the plate has an accommodation coefficient of unity, all incident particles being reflected with a Maxwellian distribution at the plate temperatures. It is further postulated that the molecular interaction is that of hard spheres.

The form of the equations that Hamel[12] obtains for this problem are written, with the following non-dimensionalization:

$$n_1' = n_1/n_\infty \qquad\qquad \Pi_{1xx}' = \Pi_{1xx}/mn_\infty V_\infty^2$$

$$N_2' = N_2/n_\infty M_\infty \qquad\qquad t' = t(V_\infty n_\infty M_\infty \pi a^2) \qquad (2)$$

$$V_1' = V_1/V_\infty \qquad\qquad x' = x(n_\infty \pi a^2 M_\infty)$$

The governing moment equations can then be written as follows:

$$\frac{\partial n_1'}{\partial t'} + \frac{\partial}{\partial x'} n_1' V_1' = n_1' N_2' |V_1'| n_2^{(0)} \frac{(\pi)^{1/2}}{2} \qquad (3)$$

$$n_1' \frac{\partial V_1'}{\partial t'} + n_1' V_1' \frac{\partial V_1'}{\partial x'} + \frac{\partial \Pi_{1xx}'}{\partial x'} = 0 \qquad (4)$$

$$\frac{\partial \Pi_{1xx}'}{\partial t'} + V_i' \frac{\partial \Pi_{1xx}'}{\partial x'} + 3\Pi_{1xx}' \frac{\partial V_1'}{\partial x'} = \frac{2(\pi)^{1/2}}{3} \times \qquad (5)$$

$$n_2^{(0)} n_1' N_2' |V_1'^3|$$

$$\frac{\partial N_2'}{\partial t} + \frac{2}{(\pi)^{1/2}} \frac{V_2(0)}{M_\infty} \frac{\partial N_2'}{\partial x} = - \frac{n_2' N_2' V_1'}{M_\infty} \qquad (6)$$

where $n_2^{(0)} = \text{erfc}[(x'/t')M_\infty]$ and $M_\infty^2 = V_1^2/(2kT_\infty/m)$.

and equations (3-5) represents the equations of motion for the "cold-fluid" and equation (6) the equation of motion for the "wall" fluid. We observe that the right hand sides of equations (3-6) represent collisions between "cold fluid" and "wall fluid" particles; so that in these equations collisions among scattered particles (e.g. "cold-cold" collisions) are neglected. To take account of these collisions we decompose the stress tensor of the cold fluid:

$$\underset{\sim}{P}_1 = p_{11} \, \underline{i}\,\underline{i} + p_\perp \left(\underline{I} - \underline{i}\,\underline{i} \right) \tag{7}$$

so that we have a component of pressure along the streamlines, p_{11}, and a component of pressure normal to streamlines, $p_\perp$. Far from the piston in the undisturbed free stream we have $p_\perp = p_{11} = p_\infty$; collisions between cold and wall particles cause anisotropy in $\underset{\sim}{P}_1$ so that $p_{11} \neq p_\perp$ in the highly rarefied regime. While in the collision-dominated regime $p_{11} = p_\perp$.

To take account of the heat flux we make an ad-hoc adjustment of the energy equation, including a heat-flow term which follows the Fourier Law. With these adjustments our new set of moment equations will go over to the Navier-Stokes equations in the continuum limit, give the correct free molecular limit and additionally give a result part way into the transition regime which will be self-consistent. In the transition regime we can consider the adjusted set of moment equations as an interpolation between a continuum and highly rarefied treatment.

We therefore rewrite equations (3-5) for the "cold-fluid" with the above adjustments:

$$\frac{d \ln n_1'}{\partial t'} + \frac{\partial n_1'}{\partial x'} = N_2' \, |V_1'| \, n_2^{(0)} \frac{(\pi)^{1/2}}{2} \tag{8}$$

$$n_1' \frac{dV_1'}{dt'} + \frac{\partial p_{11}'}{\partial x'} = 0 \tag{9}$$

$$\frac{d \ln \dfrac{p_{11}' V_1'^2}{n_1'}}{dt'} = \frac{\partial}{\partial t'} \ln \frac{V_1'^2}{n_1'} - \frac{\mu_1}{p_1'}(p_{11} - p_+) + \frac{\pi^{1/2}}{3} n_2^{(0)} N_2' V_1'^3 \tag{10}$$

6

$$\frac{d}{dt'}\left(3\,\frac{p_{11}'}{n_1'} + \frac{2p_+'}{n_1'} + V_1^{2'}\right) = \frac{1}{V_1'\,n_1'}\,\frac{\partial}{\partial x'}\,K_1\,\frac{\partial T}{\partial x'} + 2\,\frac{\partial\,p_{11}/n_1'}{\partial t}$$

(11)

where K_1 and μ_1 are the thermal conductivity and viscosity of the gas and

$\frac{d}{dt} = \frac{\partial}{\partial t'} + V_{-1}'\,\frac{\partial}{\partial x'}$. We see that for short times compared to a self-collision time between "cold-fluid" particles the above equations reduce to a near-free molecular type description:

$$\frac{d\ell n\,n_1'}{dt} + \frac{\partial}{\partial x'}\,V_1' = N_2'\,|V_1'|\,n_2^{(0)}\,\frac{\pi^{1/2}}{2}$$

(12)

$$n_1'\,\frac{dV_1'}{dt'} + \frac{\partial p_{11}'}{\partial x'} = 0$$

(13)

$$\frac{d\ell n\,\dfrac{p_{11}'\,V_1^2}{n_1'}}{dt'} = \frac{\partial}{\partial t'}\,\ell n\,\frac{V_1^2}{n_1'} + \frac{\pi}{3}\,n_2^{(0)}\,N_2'\,|V_1^3|$$

(14)

where in the right-hand sides of equations (12-14) we can substitute the free-molecular values. These equations are completely equivalent to the older model of Hamel[12] and give the near-free molecular description of the piston problem.

On the other hand, for very large time compared to a self-collision time, "wall" fluid collisions are negligible and the equations should give a steady shock. The steady state limit of the above equations can be written as:

$$n_1'\,V_1' = A$$

(15)

$$n_1'\,V_1'^2 + p_{11}' = B$$

(16)

$$\frac{d}{dx'}\,\frac{p_{11}\,V_1^{2'}}{n_1'} = -\frac{\lambda_\infty}{n_1'}\left[p_{11}' - p_\perp'\right]$$

(17)

7

$$\frac{3p'_{11}}{n'_1} + \frac{2p'_\perp}{n'_1} + V'^2_1 = \frac{K_1}{A} \frac{dT'_1}{dx'} + C \tag{18}$$

It can be shown that these equations will yield a steady state shock and this solution is now being investigated. So that the above model can be shown to have quite reasonable near-free molecular and continuum limits and at the same time give a set of equations for the transition regime that are not intractable.

In the future we expect to investigate numerically the formation of a shockwave utilizing the present work. By simply adding an electron conservation equation to the above set it may perhaps be possible to observe, theoretically the production of electrons during the formation of a shockwave.

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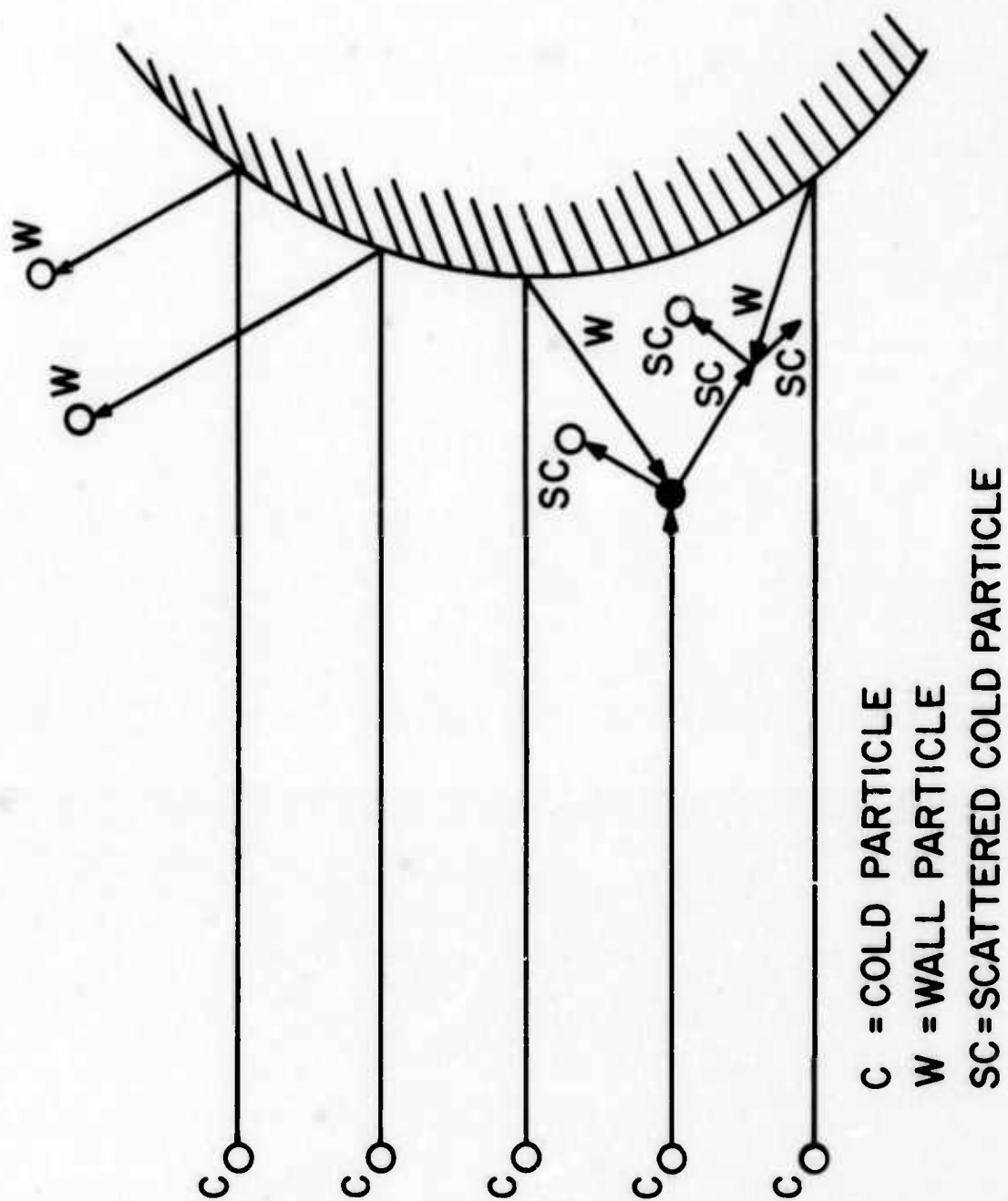13. Hamel, B. B. and Willis, D. R., "A Kinetic Theory of Source Flow Expansions," to appear June 1965.
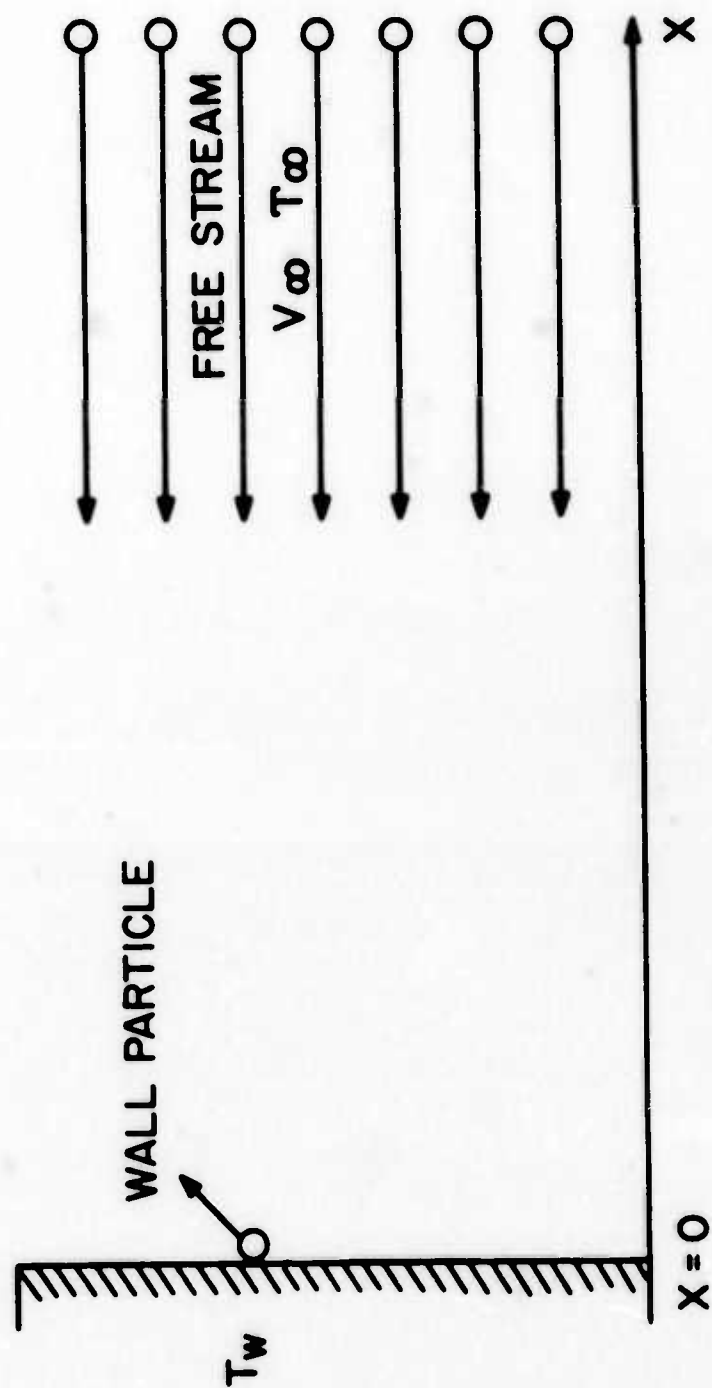
C = COLD PARTICLE
W = WALL PARTICLE
SC = SCATTERED COLD PARTICLE

Figure 1. Classification of Particles

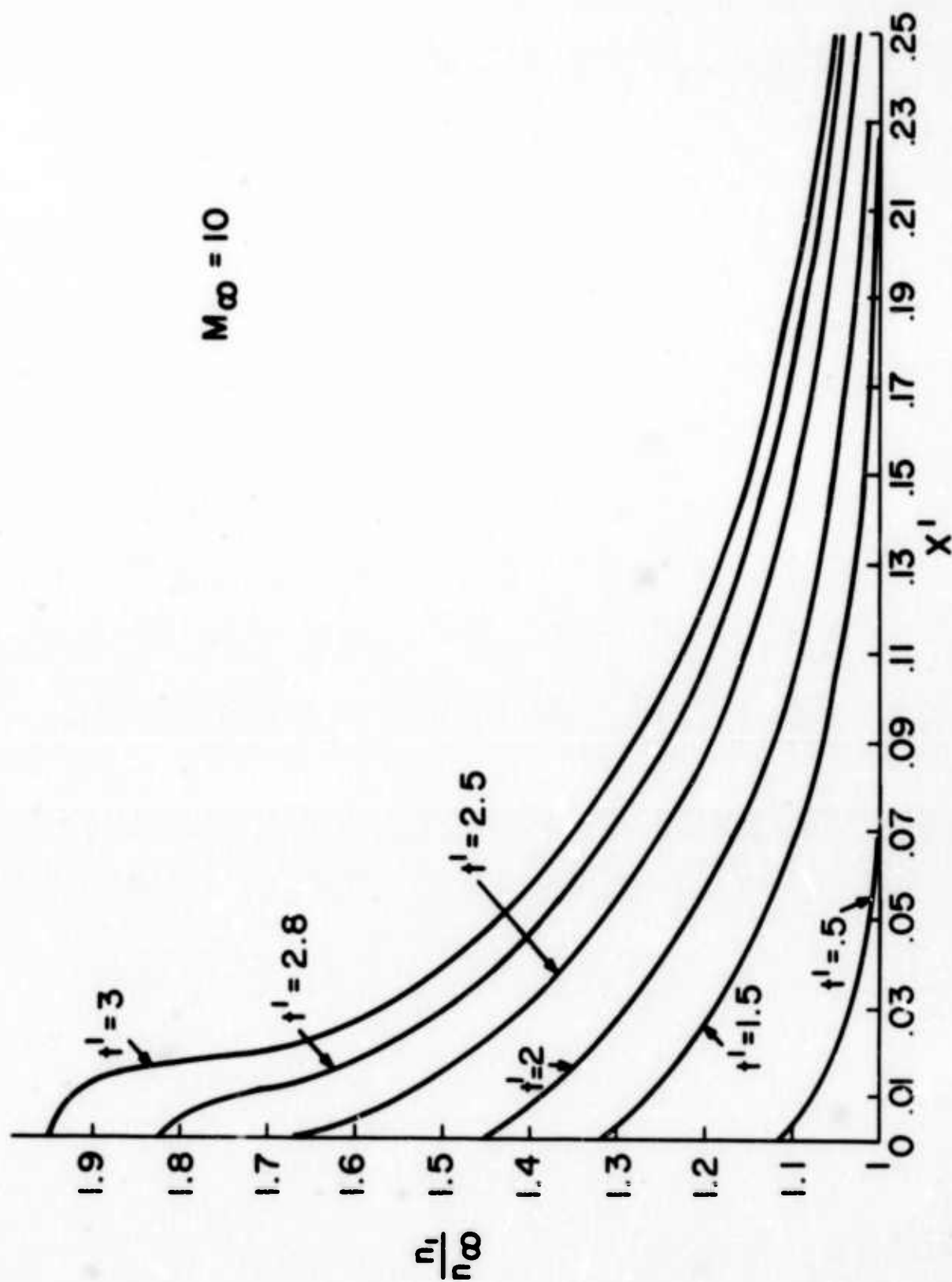Figure 2. One-Dimensional, Hypervelocity Compression

$M_\infty = 10$

Figure 3. "Cold" Fluid Number Density vs. Position

12